INFLUENCE OF SOUND WAVES ON STEADY DUCT FLOW

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The article discusses the problem of determining the secondary steady flow in a plane duct when a sound field is superimposed on an undisturbed compressible laminar flow. It is shown that under certain simplifying conditions the velocity distribution of the secondary flow in the wall region is given by a simple analytical expression. In the rest of the duct the problem is reduced to the solution of a linear fourth-order ordinary differential equation (in complex variables); this problem is solved numerically. The indicated equation is transformed to an Airy equation for large Reynolds numbers Re of the undisturbed flow. The results are presented graphically.

§1. It is well known that periodic velocity perturbations affect a steady (average) viscous fluid flow. This effect is attributable to the nonlinearity of the hydrodynamical equations. The determination of average flows is of considerable importance but meets with exceptional mathematical difficulties in the general case.

C. C. Lin [1] has introduced simplifying conditions to derive an equation for the resultant average laminar flow near the surface of a solid for the case in which steady flow is superimposed on a fluctuating fluid motion. The fluctuation components entering into the Reynolds stresses in this case can be determined independently of the steady flow.

In the present article we use Lin's idea to find the average flow in a plane duct. Suppose that sound perturbations act on an undisturbed steady laminar viscous fluid flow in such a way that the direction of the oscillatory motion of the fluid particles on the duct axis is parallel to the duct walls; we assume that the fluctuation component of the velocity on the duct axis is given by the expression

$$u_1 = A \cos kx \cdot \cos \omega t, \tag{1.1}$$

in which u_1 is the longitudinal fluctuation velocity, A is the fluctuation velocity amplitude, $k = \omega/c$ is the wave number, ω is the cyclic frequency, t is the time, c is the velocity of sound, and x is the longitudinal coordinate. The flow is considered to be two-dimensional. Gravity forces are excluded. The following assumptions are introduced:

$$\begin{cases} 1)\ \mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1,\ \rho = \rho_0 + \rho_1,\ p = p_0 + p_1; \\ 2)\ p_1 = \rho_1 c^2; \\ 3)\ \text{all variables vary significantly in the longitudinal direction at distances not less than the sound wavelength;} \\ 4)\ \text{the dynamic viscosity coefficient } \mu \text{ is constant.} \end{cases} \tag{1.2}$$

Here the subscript 0 denotes the steady-state value, the subscript 1 denotes the fluctuation value, which vanishes when averaged over a large time interval; V is the velocity; p is the pressure; and ρ is the density.

The problem is solved under the following conditions:

$$\begin{cases} 1)\ M_0 \ll 1; \\ 2)\ M_1 \ll 1; \\ 3)\ \rho_1 \ll \rho_0; \\ 4)\ \omega\lambda^2/\nu \gg 1,\ \omega h^2/\nu \gg 1, \end{cases} \tag{1.3}$$

where M is the Mach number, λ is the sound wavelength, ν is the kinematic viscosity coefficient, and h is the half-width of the duct.

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A previous attempt has been made [2] to solve this problem subject to the foregoing assumptions and conditions, but the equation obtained there for the average flow is incorrect.

§2. We adopt the following initial equations:

$$\rho[\partial \mathbf{V}/\partial t + (\mathbf{V}\nabla)\mathbf{V}] = -\nabla p + \mu\Delta\mathbf{V} + (1/3)\mu\nabla(\nabla\mathbf{V}); \qquad (2.1)$$
$$\partial\rho/\partial t + \nabla(\rho\mathbf{V}) = 0.$$

These equations must, in general, be augmented with the energy-balance and state equations, but it will be shown below that under the stated conditions and assumptions the average flow can be regarded as incompressible at distances of the order of the sound wavelength along the duct, and the fluctuation (compressible) flow is completely determined in the problem as stated.

We represent the variables $\mathbf{V}$, p, and ρ by sums of steady and fluctuation components, which we substitute into (2.1), and then after the usual operations of time averaging and subtraction of the average equations from the complete equations, rejecting small quantities by means of (1.2) and (1.3), we obtain a system of equations for the fluctuation components:

$$\rho_0\partial\mathbf{V}_1/\partial t = -\nabla p_1 + \mu\Delta\mathbf{V}_1; \qquad (2.2)$$
$$\partial\rho_1/\partial t + \rho_0\nabla\mathbf{V}_1 = 0; \; p_1 = \rho_1 c^2.$$

The last equation in (2.2) is taken from (1.2). The solution of (2.2) satisfying the sticking condition at the wall and condition (1.1) has the form [taking (1.3) into account] [3]

$$u_1 = A\cos kx[\cos\omega t - \exp(-\eta)\cdot\cos(\omega t - \eta)]; \qquad (2.3)$$
$$v_1 = -(k/\beta\sqrt{2})A\sin kx[\cos(\omega t - \pi/4) - \exp(-\eta)\cdot\cos(\omega t - \eta - \pi/4)],$$

where u_1 and v_1 are the longitudinal and transverse components of the vector $\mathbf{V}_1$; $\beta = \sqrt{\omega/2\nu}$; $\eta = y\beta$; and y is the distance from the wall. The expression for v_1 in (2.3) is valid only near the wall (at distances of order $\sqrt{\nu/\omega}$), because relations (2.3) were obtained in the boundary-layer approximation.

We now determine the steady flow. It is evident that the undisturbed steady flow can be treated as incompressible, i.e., as Poiseuille flow, at distances of the order of the sound wavelength along the duct. Thus, we approximately obtain from the expression for Poiseuille flow (allowing for the fact that $\rho_1/\rho_u \sim M_1$)

$$|\partial p_u/\partial x| \sim U_0\mu/h^2 \sim (M_u/M_1)(\nu/\omega h^2)\rho_1 c\omega,$$

where p_u, ρ_u, and M_u are the pressure, density, and Mach number in the undisturbed flow;

$$|\partial p_1/\partial x| \sim |p_1/\lambda| \sim \rho_1 c\omega.$$

Comparing the foregoing expressions, we obtain for the finite values of M_u/M_1 (or M_0/M_1)

$$|\partial p_u/\partial x| \ll |p_1/\lambda|,$$

because $\nu/\omega h^2 \ll 1$ by stipulation. Consequently, the pressure difference p_u at a distance λ along the duct is small in comparison with the fluctuation pressure difference p_1. This means that the density ρ_u as well varies only slightly at distances of order λ in comparison with the variation of the fluctuation density ρ_1, i.e., the undisturbed flow can be treated as Poiseuille flow over an interval of length λ along the duct.

We show below that the steady "increment" to the Poiseuille flow velocity is equal in order of magnitude to u_1^2/c. Then

$$|\operatorname{div}\mathbf{V}_0| \sim \frac{u_1^2}{c\lambda}. \qquad (2.4)$$

From the equation of continuity in (2.2) we obtain

$$|\operatorname{div}\mathbf{V}_1| \sim \omega\rho_1/\rho_0. \qquad (2.5)$$

It follows from (2.4) and (2.5) that

$$\frac{|\operatorname{div}\mathbf{V}_0|}{|\operatorname{div}\mathbf{V}_1|} \sim \frac{u_1^2}{\omega^2\lambda^2 M_1} \sim M_1.$$

Inasmuch as $M_1 \ll 1$ by stipulation, we have $|\operatorname{div}\mathbf{V}_0| \ll |\operatorname{div}\mathbf{V}_1|$, permitting us to set $\operatorname{div}\mathbf{V}_0 = 0$.

In light of the foregoing we write the steady-flow equation in the form

$$\rho_0(\mathbf{V}_0\nabla)\mathbf{V}_0 = -\nabla p_0 + \mu\Delta\mathbf{V}_0 - \langle\rho_1\partial\mathbf{V}_1/\partial t\rangle - \langle\rho_0(\mathbf{V}_1\nabla)\mathbf{V}_1\rangle; \qquad (2.6)$$
$$\operatorname{div}\mathbf{V}_0 = 0,$$

where the angle brackets denote the average over the wave period. The velocity $\mathbf{V}_0$ must vanish at the duct wall; moreover, it is required to specify the mass flow of fluid through the cross section of the duct (the mass flow is determined by the given Poiseuille flow). Let $\mathbf{V}_2$ denote the secondary flow velocity (u_2 and v_2 are the longitudinal and transverse components), so that the total average velocity $\mathbf{V}_0$ is the sum of $\mathbf{V}_2$ and $\mathbf{u}_0$ (Poiseuille flow velocity vector in the direction parallel to the duct wall). Then, introducing the stream function $\psi(u_2 = \partial\psi/\partial y,\ v_2 = -\partial\psi/\partial x)$ and computing the terms in the angle brackets, we rewrite (2.6) in the form

$$(u_0 + \partial\psi/\partial y)(\partial\Delta\psi/\partial x) - (\partial\psi/\partial x)(d^2u_0/dy^2 - \partial\Delta\psi/\partial y) = \nu\Delta\Delta\psi + f(y)\sin 2kx, \tag{2.7}$$

where

$$f(y) = -(A^2k\beta/2)[-2\exp(-\eta)\cos\eta - \exp(-\eta)\sin\eta + \exp(-2\eta)],$$

with the boundary conditions

at the wall ($y=0$)

$$\psi = \partial\psi/\partial y = 0;$$

on the axis ($y=h$)

$$\psi = \partial^2\psi/\partial y^2 = 0.$$

The function $f(y)$ differs significantly from zero only in a wall zone of thickness $\sim\sqrt{\nu/\omega}$. It can therefore be assumed that the unknown secondary flow forms a boundary layer in the wall zone (this assumption is justified by the end result). Then, comparing the terms on the left-hand side of Eq. (2.7) with the term $\nu\Delta\Delta\psi$, we verify that they are all much smaller in absolute value than the comparison term. For example, the first term in (2.7) has order of magnitude

$$|u_0\partial\Delta\Psi/\partial x| \sim |u_0\psi|\,\delta^2\lambda,$$

where $\delta = \sqrt{2\nu/\omega}$; also

$$|\nu\Delta\Delta\psi| \sim |\nu\psi|\,\delta^4 \sim |\omega\psi|/\delta^2.$$

It is evident from this result that

$$\frac{|u_0\partial\Delta\psi/\partial x|}{|\nu\Delta\Delta\Psi|} \sim u_0/\lambda\omega \sim M_0.$$

The given ratio is much smaller than unity, because $M_0 \ll 1$ by stipulation. Thus, in a wall zone of thickness $\sim\delta$ Eq. (2.7) is simplified:

$$\nu(\partial^4\psi/\partial y^4) = -f(y)\sin 2kx. \tag{2.8}$$

The solution of (2.8) satisfying the wall sticking condition and bounded in the wall zone has the form

$$u_2 = (A^2k\sin 2kx/4\nu\beta^2)[-(1/2)\exp(-\eta)\cos\eta - (3/2)\exp(-\eta)\sin\eta - -(1/4)\exp(-2\eta) + 3/4], \tag{2.9}$$

where $u_2 = \partial\psi/\partial y$. We do not compute the expressions for the transverse component v_2, because it is not needed below.

Substituting (2.9) into (2.7), we verify that all terms on the left-hand side of (2.7) are less in absolute value than $|\nu\Delta\Delta\psi|$. At the outer boundary of the wall zone ($\eta\to\infty$) we have from (2.9)

$$u_2 = (3/8)(A^2/c)\sin 2kx.$$

In the external domain, i.e., everywhere except the wall zone of thickness $\sim\delta$, the following equation holds:

$$(u_0 + \partial\psi/\partial y)(\partial\Delta\psi/\partial x) - (\partial\psi/\partial x)(d^2u_0/dy^2 + \partial\Delta\psi/\partial y) = \nu\Delta\Delta\psi, \tag{2.10}$$

along with the boundary conditions

$$\psi=0,\ \partial\psi/\partial y = (3/8)(A^2/c)\sin 2kx \quad \text{at} \quad y=0;$$
$$\psi=0,\ \partial^2\psi/\partial y^2=0 \quad \text{at} \quad y=h. \tag{2.11}$$

The error in [2] arises insofar as Eq. (2.8) is assumed to be valid in the wall and the external domains; in the present statement of the problem there is no justification for this inference.

It is evident from (2.10) and (2.11) that the value of the function $\partial\psi/\partial y = u_2$ is equal in order of magnitude to A^2/c, whence we infer that

$$|u_2/U_0| \sim A^2/cU_0 = (M_1/M_0)^2 M_0, \tag{2.12}$$

where U_0 is the maximum value of the Poiseuille flow velocity. For finite values of $(M_1/M_0)^2$ the ratio of $|u_2|$ to $|U_0|$ is small, since $M_0 \ll 1$. The characteristic linear scales of problem (2.10), (2.11) are the sound wavelength λ and the half-width h of the duct. We can therefore estimate the terms of Eq. (2.10), showing that the nonlinear terms on the left-hand side are small in comparison with the linear terms; the given equation is then linearized and assumes the form

$$u_0 \partial \Delta\psi/\partial x - (d^2u_0/dy^2)(\partial\psi/\partial x) = \nu\Delta\Delta\psi. \tag{2.13}$$

Relations (2.13) and (2.11) describe the secondary flow in the external domain, i.e., everywhere except a narrow wall zone of thickness $\sim\delta$, in which the secondary longitudinal velocity u_2 is given by expression (2.9).

We seek the solution of problem (2.13), (2.11) in the form

$$\psi(x, y) = \text{Real } \Phi(y) \exp(-2ikx). \tag{2.14}$$

We introduce the following dimensionless (indicated by an overbar) variables:

$$\bar{\Phi} = 8\Phi c/3hA^2; \ \bar{y} = y/h; \ \bar{u}_0 = u_0/U_0.$$

Substituting (2.14) into (2.13) and (2.11), we obtain a linear ordinary differential equation in $\bar{\Phi}$:

$$\bar{\Phi}'''' - \bar{\Phi}''(8\alpha^2 - 2i\alpha\text{Re}\bar{u}_0) + \bar{\Phi}(16\alpha^4 + 4i\alpha\,\text{Re} - 8i\alpha^3\,\text{Re}\,\bar{u}_0) = 0 \tag{2.15}$$

along with the boundary conditions

$$\bar{\Phi} = 0, \ \bar{\Phi}' = i \ \text{ for } \bar{y} = 0; \qquad \bar{\Phi} = \bar{\Phi}'' = 0 \ \text{ for } \bar{y} = 1, \tag{2.16}$$

where the prime denotes differentiation with respect to $\bar{y}$; α = kh; Re = $U_0 h/\nu$.

Problem (2.15), (2.16) is readily solved by a numerical method for not too large values of the parameter Re α. The linear boundary-value problem is reduced to a system of Cauchy problems [4], which can be integrated, for example, by the Runge–Kutta method.

The results of the solution of problem (2.15), (2.16) are given in Figs. 1-3. Figure 1 shows the secondary-flow streamline pattern for Re = 50, α = 1, where the zero ordinate corresponds to the outer boundary of the wall boundary layer of thickness $\sim\delta$ and unity, to the middle of the duct. Figures 2 and 3 show the profiles of the dimensionless longitudinal velocity $\bar{u}_2 = 8u_2c/3A^2$ in various cross sections of the duct for Re = 50 and 1000, respectively (solid curves) and α = 1. The $\bar{u}_2$ profiles in the duct cross sections corresponding to 2kx > π are not shown, because in this case $\bar{u}_2$ changes sign, i.e., $\bar{u}_2(2kx + \pi) = -\bar{u}_2(2kx)$ according to (2.14).

§3. For large values of Re α difficulties arise in the numerical computation of problem (2.15), (2.16) in connection with the smallness of the parameter 1/Re α for the leading derivative in Eq. (2.15). However, the problem can be simplified. For Re $\alpha \gg 1$ the secondary flow also forms a boundary layer in the external domain (as is evident from Figs. 2 and 3). Within the limits of this boundary layer the expression for the Poiseuille flow velocity can be regarded as a linear function of the transverse coordinate. With the foregoing in mind we can compare the terms in Eq. (2.15) as in the derivation of the Prandtl equations; after the rejection of small terms Eq. (2.15) assumes the form (in new dimensionless variables)

$$\tau'' - \eta\tau = 0, \tag{3.1}$$

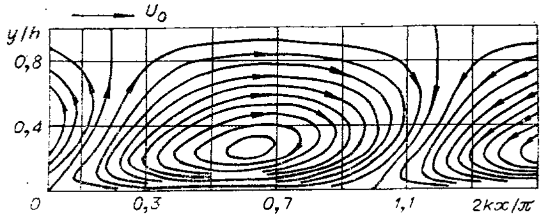

Fig. 1

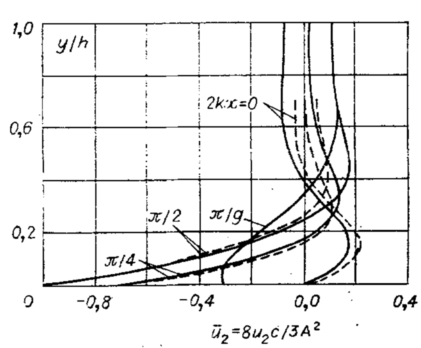

Fig. 2

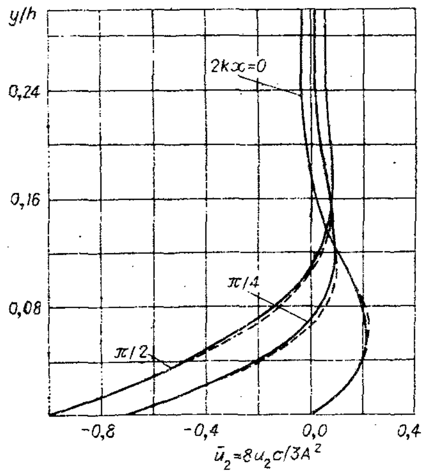

Fig. 3

where the prime denotes differentiation with respect to the coordinate η:

$$\eta = (-4i)^{1/3} \frac{y\,(\mathrm{Re}\alpha)^{1/3}}{h}; \quad \tau = \overline{\Phi}''; \quad \overline{\Phi} = \frac{\Phi\,(\mathrm{Re}\alpha)^{1/3}(-4i)^{1/3}c}{(3/8)hA^2}.$$

The function $\overline{\Phi}$ and its derivatives are subject to the boundary conditions

$$\overline{\Phi}=0,\ \overline{\Phi}'=i \ \text{for} \ \eta=0; \qquad \overline{\Phi}'=\overline{\Phi}''=0 \ \text{for} \ |\eta| \to \infty. \tag{3.2}$$

Equation (3.1) is an Airy equation. The dashed curves in Figs. 2 and 3 represent the longitudinal velocity profiles $\bar{u}_2$ obtained from (3.1) and (3.2), along with the same results reduced to the corresponding values of Re and α. It is seen that for Re α = 1000 (see Fig. 3) the solutions of problems (2.15), (2.16) and (3.1), (3.2) differ insignificantly.

Equation (3.1) is valid under definite constraints on the values of Re α. The linear equation (2.15) was obtained from (2.10) by means of the approximate relation (2.12). In the case of large values of Re α, however, the maximum Poiseuille flow velocity U_0 cannot be used as the characteristic velocity scale of the undisturbed flow (2.12), because the thickness δ_e of the boundary layer becomes small in comparison with the width of the duct. In this case the appropriate velocity scale is the quantity $U_0\,\delta_e/h$. For the linearization of Eq. (2.10) it is necessary to satisfy the condition [analogous to condition (2.12)]

$$|u_2 h/U_0\delta_e| \ll 1. \tag{3.3}$$

Inasmuch as

$$\delta_e/h \sim 1/(\mathrm{Re}\,\alpha)^{1/3},$$

we can rewrite (3.3) in the form

$$(\mathrm{Re}\,\alpha)^{1/3} \ll (M_0/M_1)^2(1/M_0).$$

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